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1983 J. Phys. A: Math. Gen. 16 3639

(http://iopscience.iop.org/0305-4470/16/15/029)

J. Phys. A: Math. Gen. 16 (1983) 3639–3643. Printed in Great Britain

# On SU(3) monopoles in the Yang $R$-gauge

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Received 21 April 1982

**Abstract.** The self-duality equations for the static SU(3) Yang–Mills–Higgs system in the $R$-gauge has been reduced to a set of coupled ordinary differential equations, by means of a one-function ansatz. The SU(2) embedding solutions are recovered.

The Yang $R$-gauge (Yang 1977) has proved to be a very useful tool in the search for exact solutions to the Yang–Mills field equations (Corrigan *et al* 1978). As it happens, the method of the $R$-gauge turns out to be more useful in the case of static monopole solutions to the Yang–Mills–Higgs system than it is for the pure Yang–Mills, and is implicit in Ward's (1981) construction of SU(2) monopole solutions. Subsequently, Prasad (1981) has highlighted the role of the $R$-gauge in the construction of SU(2) multimonopoles.

It is therefore natural to ask whether the Yang $R$-gauge plays a similarly useful role in the search for SU(3) self-dual monopoles. It is this task that we address ourselves to in the present article. Recently, Ward (1981) has found a class of SU(3) monopole solutions without recourse to the $R$-gauge method.

In a recent paper (Singh and Tchrakian 1981), the self-dual Yang–Mills potentials were parametrised in terms of two real, $\phi_1$, $\phi_2$, and three pairs of complex, $\rho_1$, $\bar\rho_1$, $\rho_2$, $\bar\rho_2$, $\rho_3$, $\bar\rho_3$, functions of the complex variables $y$, $\bar y=(1/\sqrt{2})(x_1\pm ix_2)$; $z$, $\bar z=(1/\sqrt{2})(x_3\pm ix_4)$. These potentials in the $R$-gauge will have real values for their Cartesian components for real values of $x_\mu$ ($\mu=1\ldots4$), provided that the following conditions are satisfied

$$\bar\rho_i \doteq \rho_i^*, \qquad i=1,2,3, \tag{1}$$

with the notation of Yang (1977) for $\doteq$.

Here we seek solutions to the self-duality equations ($10a,b$), ($11a,b$), (12), (13), (14) and (15) of Singh and Tchrakian (1981), satisfying (a) a one-function ansatz, and (b) boundary conditions suitable for a monopole solution.

(a) We start with the assumption that each of the above named functions that parametrise the SU(3) Yang–Mills potentials depends on $y$, $\bar y$, $z$, $\bar z$ through the *single* function $f(y,\bar y,z,\bar z)$, which is subject to

$$f_{y\bar y}+f_{z\bar z}=0. \tag{2}$$

Our ansatz is then stated as follows:

$$\left.\begin{aligned}\bar\rho_{2\bar y}-\bar\rho\bar\rho_{1\bar y}&=\theta(f)f_z\\ \bar\rho_{2\bar z}-\bar\rho\bar\rho_{1z}&=-\theta(f)f_y\end{aligned}\right\},\qquad \left.\begin{aligned}\rho_{2y}-\rho\rho_{1y}&=\bar\theta(f)f_{\bar z}\\ \rho_{2z}-\rho\rho_{1z}&=-\bar\theta(f)f_{\bar y}\end{aligned}\right\} \tag{3a, b}$$

$$\left.\begin{aligned}\bar{\rho}_{1\bar{y}} &= \psi(f) f_z \\ \bar{\rho}_{1\bar{z}} &= -\psi(f) f_y\end{aligned}\right\}, \qquad \left.\begin{aligned}\rho_{1y} &= \bar{\psi}(f) f_{\bar{z}} \\ \rho_{1z} &= -\bar{\psi}(f) f_{\bar{y}}\end{aligned}\right\} \tag{4a, b}$$

where $\rho = \rho_3/\phi_1$, $\bar{\rho} = \bar{\rho}_3/\phi_1$, and, $\theta$, $\bar{\theta}$, $\psi$, $\bar{\psi}$ are some functions of $f$, as yet to be determined, and satisfy, due to condition (1), the equations

$$\bar{\theta}(f) \doteq \theta^*(f), \qquad \bar{\psi}(f) \doteq \psi^*(f). \tag{1'a, b}$$

It is immediately seen that $(3a, b)$ and $(4a, b)$ solve, respectively, equations $(10a, b)$ and $(11a, b)$ of Singh and Tchrakian (1981).

Denoting the derivatives of all functions of $f$ with respect to $f$ as $\dot{\phi}_1 = d\phi_1/df$, $\dot{\rho}_1 = d\rho_1/df$, etc, we write down the consequences of our one-function ansatz. First we learn from $(3a, b)$, $(4a, b)$ and (2) that $\psi = c$ and $\bar{\psi} = \bar{c}$, where $c$ and $\bar{c}$ are constants, conjugate complex to each other, and that

$$\dot{\theta} + c\dot{\bar{\rho}} = 0, \qquad \dot{\bar{\theta}} + \bar{c}\dot{\rho} = 0. \tag{5a, b}$$

It then follows from our ansatz that the remaining self-duality equations, namely (12)–(15) of Singh and Tchrakian (1981) reduce to the following set of coupled nonlinear ordinary differential equations in the variable $f$:

$$\mathrm{d}^2/\mathrm{d}f^2 \ln \phi_1 + \frac{|c|^2}{\phi_1^2} + \frac{1}{2}\frac{|\theta|^2}{\phi_2^2} - \frac{1}{2}\left(\frac{\phi_1}{\phi_2}\right)^2 \frac{|\dot{\theta}|^2}{|c|^2} = 0, \tag{6}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}f^2} \ln \phi_2 + \frac{1}{2}\frac{|c|^2}{\phi_1^2} + \frac{|\theta|^2}{\phi_2^2} + \frac{1}{2}\left(\frac{\phi_1}{\phi_2}\right)^2 \frac{|\dot{\theta}|^2}{|c|^2} = 0, \tag{7}$$

$$\ddot{\theta} + \dot{\theta}(\mathrm{d}/\mathrm{d}f) \ln(\phi_1/\phi_2)^2 - (c/\phi_1^2)\theta = 0, \tag{8a}$$

$$\ddot{\bar{\theta}} + \dot{\bar{\theta}}(\mathrm{d}/\mathrm{d}f) \ln(\phi_1/\phi_2)^2 - (\bar{c}/\phi_1^2)\bar{\theta} = 0. \tag{8b}$$

A solution to these equations would yield the functions $\phi_1$, $\phi_2$, $\theta$, $\bar{\theta}$ in terms of $f$. The last step in this systematic procedure would be the integration, with respect to $y$, $\bar{y}$, $z$, $\bar{z}$, of $(3a, b)$, $(4a, b)$ and $(5a, b)$ to yield $\rho_1$, $\bar{\rho}_1$, $\rho_2$, $\bar{\rho}_2$, and $\rho_3$, $\bar{\rho}_3$ respectively.

(b) Next, we must make sure that the integration of (6)–(8) should give rise to solutions that exhibit the appropriate behaviour for a monopole solution. This requirement will impose further conditions that the solutions must satisfy,which are given below.

Following the procedures of Ward (1981) and Prasad (1981), we perform the dimensional reduction leading to the static ($x_4$-independent) Yang–Mills–Higgs system with the fourth component of the potential identified as the Higgs field $\Phi$. This we do by attributing the following explicit $x_4$-dependences to the $R$-gauge parameters:

$$\begin{aligned}
&\phi_1(x) = \mathring{\phi}_1(\boldsymbol{x}) \exp(\mathrm{i}\lambda_1 x_4), \\
&\qquad \rho_1(x) = \mathring{\rho}_1(\boldsymbol{x}) \exp(\mathrm{i}\lambda_4 x_4), \qquad \bar{\rho}_1(x) = \mathring{\bar{\rho}}_1(\boldsymbol{x}) \exp(\mathrm{i}\lambda_1 x_4),
\end{aligned} \tag{9}$$

$$\begin{aligned}
&\phi_2(x) = \mathring{\phi}(\boldsymbol{x}) \exp(\mathrm{i}\lambda_2 x_4), \\
&\qquad \rho_2(x) = \mathring{\rho}_2(\boldsymbol{x}) \exp(\mathrm{i}\lambda_2 x_4), \qquad \bar{\rho}_2(x) = \mathring{\bar{\rho}}_2(\boldsymbol{x}) \exp(\mathrm{i}\lambda_2 x_4), \\
&\qquad \rho_3(x) = \mathring{\rho}_3(\boldsymbol{x}) \exp(\mathrm{i}\lambda_2 x_4), \qquad \bar{\rho}_3(x) = \mathring{\bar{\rho}}_3(\boldsymbol{x}) \exp(\mathrm{i}\lambda_2 x_4),
\end{aligned} \tag{10}$$

which result in $x_4$-independent potentials $A_\mu$, cf equations $(6a, b)$ of Singh and Tchrakian (1981). It is then possible to express the square of the magnitude of the Higgs field in terms of the derivatives with respect to $y$, $\bar{y}$, $z$, $\bar{z}$ of the $R$-gauge parameters:

$$\|\Phi\|^2 = \tfrac{2}{3}(\lambda_1^2+\lambda_2^2-\lambda_1\lambda_2)+\frac{4}{3}\frac{\phi_{1z}}{\phi_1}\frac{\phi_{1\bar z}}{\phi_1}+\frac{4}{3}\frac{\phi_{2z}}{\phi_2}\frac{\phi_{2\bar z}}{\phi_2}-\frac{2}{3}\frac{\phi_{1z}}{\phi_1}\frac{\phi_{2\bar z}}{\phi_2}$$
$$-\frac{2}{3}\frac{\phi_{2z}}{\phi_2}\frac{\phi_{1\bar z}}{\phi_1}+\frac{\rho_{1z}\bar\rho_{1\bar z}}{\phi_1^2}+\frac{(\rho_{2z}-\rho\rho_{1z})(\bar\rho_{2\bar z}-\bar\rho\bar\rho_{1\bar z})}{\phi_2^2}+\left(\frac{\phi_1}{\phi_2}\right)^2\rho_z\bar\rho_{\bar z}. \tag{11}$$

We are now in a position to impose the behaviour required of $\|\Phi\|^2$ for a suitable monopole solution:

$$\|\Phi\|^2 = \text{constant} - \mu\Box\ln f. \tag{12}$$

where the coefficient of the $1/r$ term in the expansion of $\|\Phi\|^2$ is related to the topological charge of the solution (Ward 1981).

From (11) and (12) then follow the two additional conditions to be satisfied by the solutions of (6)–(8):

$$\frac{4}{3}\left[\left(\frac{\dot\phi_1}{\phi_1}\right)^2+\left(\frac{\dot\phi_2}{\phi_2}\right)^2-\left(\frac{\dot\phi_1}{\phi_1}\right)\left(\frac{\dot\phi_2}{\phi_2}\right)\right]+\left(\frac{\phi_1}{\phi_2}\right)^2\frac{|\dot\theta|^2}{|c|^2}=\frac{\mu}{f^2}, \tag{13}$$

$$\frac{|c|^2}{\phi_1^2}+\frac{|\theta|^2}{\phi_2^2}=\frac{\mu}{f^2}. \tag{14}$$

Finally, since we are interested in the static solutions of the Yang–Mills–Higgs system, we must determine the explicit $x_4$-dependence of the function $f$ in terms of which all the $R$-gauge parameters are expressed. For this we must first integrate (6)–(8) with respect to $f$, but this we have not yet done. Fortunately however, substituting (14) into (6) and (7) and adding, leads, after integration, to

$$\phi_1\phi_2 = f^{3\mu/2} \tag{15}$$

according to which it follows that the $x_4$ dependence of $f$ is

$$f(x_\mu) = \mathring{f}(\boldsymbol{x})\exp\{\tfrac{2}{3}\mathrm{i}[(\lambda_1+\lambda_2)/\mu]x_4\} = \mathring{f}(\boldsymbol{x})\exp(\mathrm{i}\lambda x_4). \tag{16}$$

It then follows from (2) that $\mathring{f}$ satisfies the Helmholtz equation with a spherically symmetric solution

$$\mathring{f}(\boldsymbol{x}) = (\sinh\lambda r)/r. \tag{17}$$

In an attempt to integrate equations (6)–(8) completely, we eliminate $\phi_2$ from (6) by substituting for $\phi_2$ from (15), which gives

$$\frac{\ddot\phi_1}{\phi_1}+\left(\frac{\dot\phi_1}{\phi_1}\right)^2-\frac{3\mu}{f}\left(\frac{\dot\phi_1}{\phi_1}\right)+\frac{3\mu^2}{2f^2}+\frac{1}{2}\frac{|c|^2}{\phi_1^2}=0. \tag{18}$$

Denoting $g=\phi_1^2/|c|^2$, (18) can be put into the form

$$f^2\ddot g - 2\mu f\dot g + 3\mu^2 g = -f^2 \tag{18$'$}$$

which is an equation of Euler type with particular integral

$$g=\alpha f^2, \qquad \alpha^{-1} = -2+6\mu-3\mu^2, \tag{19}$$

and whose homogeneous part has general solution

$$g = Af^{n_1}+Bf^{n_2} \tag{19$'$}$$

with $A$ and $B$ integration constants, and $n_1$ and $n_2$ the roots of

$$n^2-(1+3\mu)n+3\mu^2=0. \tag{20}$$

The general solution for $\phi_1$ is then

$$\phi_1=|c|(Af^{n_1}+Bf^{n_2}+\alpha f^2)^{1/2}. \tag{21}$$

Having found $\phi_1$ and $\phi_2$, it is now in principle possible to solve the linear second-order homogeneous equations for $\theta$ and $\bar{\theta}$ in terms of $f$. Then the two constants arising from this last integration, along with $A$, $B$ and $\alpha$, will have to be chosen so that (13) and (14) are also satisfied.

This is a rather complicated task and we have only succeeded in recovering SU(2) embedding solutions and one which violates the reality condition (5). We present these below.

(i) $\theta=a$, $\bar{\theta}=\bar{a}$, $a$ and $\bar{a}=a^*$ constants. Then (8) lead to $\theta=\bar{\theta}=0$, and (14) leads to $\phi_1=(|c|/\sqrt{\mu})f$. But (6) and (7) give $\phi_1=f^{\mu}$ and $\phi_2=f^{\mu/2}$, and therefore the only choice for $\mu$ is $\mu=1$. The solution automatically satisfies (13).

Then, from (5) it follows that $\rho$ and $\bar{\rho}$ are constants and hence this solution only involves the parameters $\phi_1$, $\phi_2$ and $\rho_1$, $\bar{\rho}_1$.

We now insert the parameters of this solution into the Higgs field

$$\mathrm{i}\sqrt{2}A_4=\begin{bmatrix} -(\sqrt{2}/3)\partial_3\ln\phi_1\phi_2 & \bar{\rho}_{1\bar{z}}/\phi_1 & (\bar{\rho}_{2\bar{z}}-\bar{\rho}\bar{\rho}_{1\bar{z}})/\phi_2 \\ \rho_{1z}/\phi_1 & -(\sqrt{2}/3)\partial_3\ln\phi_2\phi_1^{-2} & (\phi_1/\phi_2)\bar{\rho}_{\bar{z}} \\ (\rho_{2z}-\rho\rho_{1z})/\phi_2 & (\phi_1/\phi_2)\rho_z & -(\sqrt{2}/3)\partial_3\ln\phi_1\phi_2^{-2} \end{bmatrix} \tag{22}$$

which then reduces to

$$A_4=\mathrm{i}\begin{bmatrix} \frac{1}{2}\partial_3\ln f & (c/\sqrt{2})\partial_y\ln f & 0 \\ (\bar{c}/\sqrt{2})\partial_{\bar{y}}\ln f & -\frac{1}{2}\partial_3\ln f & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

clearly, an SU(2) embeding.

(ii) $\psi=0$, $\bar{\psi}=0$, implying through (5) that $\theta=k$, $\bar{\theta}=\bar{k}=k^*=$ constant. Then (14) leads to $\phi_2=(|k|/\mu)f$, and, from (6) and (7) it follows that $\phi_1=(\sqrt{\mu}/|k|)f^{3\mu/2-1}$. On the other hand (8$a$, $b$) imply $\dot{\rho}=l(\phi_2/\phi_1)^2$, $\dot{\bar{\rho}}=\bar{l}(\phi_2/\phi_1)^2$ where $l$, $\bar{l}$ are constants, complex conjugate to each other. Substituting these for $|\dot{\rho}|^2$ into (6) or (7) leads to one of the following two restrictions.

(1) $\mu=2$, which is a non-SU(2) embedding solution as seen from (22). But this solution is subject to the final condition $|k|^2|\theta|^2+2=0$, which violates the reality condition (5) and hence is not interesting.

(2) $\mu=1$ and $|k|=0$ and therefore $|\dot{\rho}|=0$. Condition (13) is automatically satisfied (as in (1) above). In this case the potentials in the $R$-gauge are parametrised by $\phi_1$, $\phi_2$ and $\rho_2$, $\bar{\rho}_2$. Inserting these into (22) yields

$$A_4=\mathrm{i}\begin{bmatrix} \frac{1}{2}\partial_3\ln f & 0 & (c/\sqrt{2})\partial_y\ln f \\ 0 & 0 & 0 \\ (\bar{c}/\sqrt{2})\partial_{\bar{y}}\ln f & 0 & -\frac{1}{2}\partial_3\ln f \end{bmatrix}$$

which is obviously another SU(2) embedding.

It appears therefore that neither $\theta$ nor $\psi$ can vanish if we wish to find a non-SU(2) embedding solution. In this case the expressions become very complicated, and simple solutions do not seem to work. For example, if we considered the special case of (21)

$$\phi_1 = |c|\sqrt{\alpha} f, \qquad A = B = 0, \tag{21'}$$

the solutions of $(8a, b)$ are

$$\theta = f^{3/2(\mu-1)}(A'f^n + B'f^{-n}), \tag{23a}$$

$$\bar{\theta} = f^{3/2(\mu-1)}(\bar{A}'f^n + \bar{B}'f^{-n}), \qquad n = (1 + 6\mu - 3\mu^2)^{1/2}, \tag{23b}$$

and consistency with (14) leads to $B' = \bar{B}' = 0$, $\mu = 1 \pm (2/\sqrt{3})$ and $|c^2|A'\bar{A}' + 6(2 \pm 1/\sqrt{3}) = 0$. This last condition cannot be satisfied without violating the reality conditions (1).

The usefulness of the Yang $R$-gauge method in the search for SU(3) monopole solutions depends on being able to integrate the equations (6)–(8) without violating the reality conditions. So far we have only succeeded in recovering the two SU(2) embedding solutions.

## Acknowledgment

I am grateful to Dr L P Singh for numerous discussions at the initial stages of this work.

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